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Technical Notes

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Enhancement of Stability of Composite Plates Using Shape Memory Alloy Supports

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Nomenclature

a, b	=	dimensions of the plate
D_{ij}	=	bending stiffnesses
k_i	=	stiffness of equivalent elastic foundation provided by a system of parallel SMA wires
l_i	=	spacing of SMA wires oriented in the jth direction $(i, j = x, y)$
m, n	=	integer numbers
N_x, N_y	=	applied stress resultants
p	=	counterpressure applied by a system of SMA wires connected to the plate at a number of points
T	=	recovery force in a SMA wire
x_i	=	coordinate of a SMA wire oriented in the y direction
y_j	=	coordinate of a SMA wire oriented in the x direction

I. Introduction

SHAPE memory alloy (SMA) fibers and wires generating tensile forces as a result of the phase transformation have been considered for the enhancement of stiffness of composite and metallic plates by numerous investigators [1]. Two typical methods are direct embedding of SMA fibers bonded to the substrate material and placing such fibers in sleeves where they can slide without a direct contact with the substrate (the latter method was proposed by Baz et al. [2]). Both these methods result in an effective strengthening of the structure, as long as the boundaries are capable of carrying reaction forces from the fibers produced by the phase transformation. Although early work using both methods referred to above has been reviewed in [1], recent studies concerned with both static and dynamic problems are represented by papers [3–5]. Various aspects related to the stability of SMA-reinforced plates were analyzed in [3,4,6–8].

Disadvantages of placing SMA fibers within the structure include technical problems related to their activation, the issues of integrity of the structure affected by the sleeves and SMA fibers, and the effect of an elevated temperature in activated fibers on the properties of the substrate. These shortcomings can be overcome by bonding the sleeves with the fibers to the surface of the structure. Such a method was suggested by Epps and Chandra [9] in their study of free vibrations of composite beams that showed that prestressed SMA wires in sleeves externally bonded to the structure serve as an equivalent elastic foundation resisting deflections. In the recent study, the author illustrated that SMA wires in sleeves strategically bonded to a composite or isotropic plate form a nonuniform elastic mesh foundation that can effectively reduce the amplitude of forced vibrations [10]. The other efficient method of reducing forced vibrations that was investigated in the same paper is based on supporting the plate by SMA wires connected to the plate at selected points. The present Note expands the possible range of applications of both methods that is limited to the problem of forced vibrations in [10] by analyzing their effectiveness for the enhancement of stability of composite plates.

II. Composite Plate Supported by an Elastic Foundation Provided by SMA Wires

Consider a thin, simply supported, symmetrically laminated rectangular plate with SMA wires in sleeves bonded to one of the surfaces within the area $a_1 < x < a_2$, $b_1 < y < b_2$ and subject to biaxial compression as shown in Fig. 1. SMA wires can freely slide along the sleeves, that is, friction is negligible. Although the wires are not in direct contact with the plate, they bend together with the plate and the sleeves. The reaction of the wires is transferred to the plate in the form of counterpressure resisting bending. However, the axial tensile force in the wires is not transferred to the plate; it is balanced by the supporting boundary as explained in detail in the following discussion on the boundary conditions. It is assumed that the stiffness of the sleeve material is small, that is, the sleeves do not affect the matrix of stiffnesses of the plate that remains symmetric. If necessary, this requirement can be enforced by cutting the sleeves at periodic, closely spaced intervals. Before installation, a SMA wire is cooled so that the material is transformed into martensite and stretched to a required length. Subsequently, it is inserted in the sleeves and constrained by joining it to the supporting structure. As temperature returns to the operational level corresponding to the austenitic phase of SMA, the wire is in tension as a result of constrained recovery. The spacing between the sleeves can be arbitrary, that is, it is possible to selectively increase the stiffness of the central part of the plate. This enables us to maximize the effect of SMA wires with a minimum additional weight to the plate. With the wires being outside of the plate, the heat transfer to the plate during their activation is limited. Hence, it is possible to assume that the activation temperature in the wires does not result in a local reduction of stiffness of the composite material. It is also assumed that thermal contributions associated with temperature responsible for the activation of SMA wires do not affect equations of equilibrium.

It is important to clarify boundary conditions relevant to the approach discussed in the previous paragraph. Obviously, if the boundary supporting both the plate and the SMA wires is movable in-plane, the reaction of wires is transferred to the plate and all advantages of the suggested approach are lost. In an experimental setup, the plate and wires could have separate boundaries; thus the reaction of wires would not detrimentally affect the stability of the plate. In realistic structures the problem considered here would occur if the plate with immovable boundaries were subject to a uniform temperature. Then prebuckling thermally induced compressive stress resultants can easily be calculated. Interestingly, the applied

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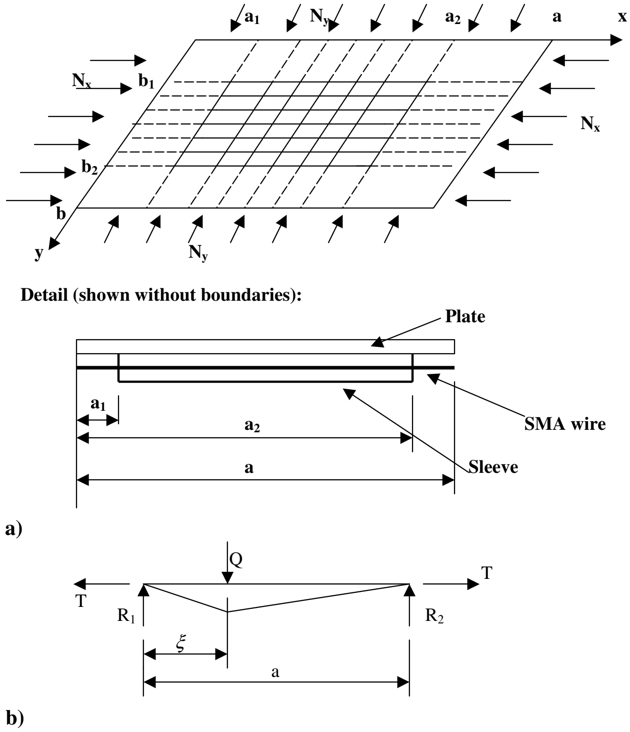

Fig. 1 a) The plate supported by SMA wires in sleeves within the regions $a_1 < x < a_2$, $b_1 < y < b_2$. The solid lines identify wires in the region of the plate supported by the sleeves. The dashed lines are wires in the unsupported regions; b) the free body diagram of a wire oriented in the x direction including the applied force and reactions.

temperature could also trigger the reverse phase transformation in SMA wires, eliminating the need in a separate power supply. Note that in-plane "immovable" boundaries can be found in applications where identical adjacent plates are subject to identical loading. In such a case, boundary in-plane displacements of adjacent plates "cancel out" each other irrespective of the out-of-plane stiffness of the supports.

Another example of the problem relevant to this study is found in plates supported by immovable boundaries if the latter experience a prescribed in-plane displacement (such a situation is also observed in experimental setups). Although the distance between the boundaries is reduced as a result of this displacement, this change is too small to noticeably affect the restoring force generated in SMA wires as a result of the phase transformation. Accordingly, the linear analysis of the corresponding buckling problem can be conducted using a straightforward modification of the present solution.

The response of a single wire to deflections of the plate can be analyzed by assuming that its bending stiffness is negligible. Consider, for example, a wire oriented in the x direction and subject to a concentrated load Q applied at $x = \xi$ as shown in Fig. 1b. The deflection of the wire at the point of application of the force is available from the requirement that the bending moment at $x = \xi$ must be equal to zero:

$$w(\xi) = \frac{Q(a-\xi)\xi}{Ta} \tag{1}$$

Accordingly, the local stiffness of the system of wires oriented in the x direction is available in the form (also, see [9,10])

$$k_2(x, y) = \frac{T}{(a-x)x}\delta(y - y_j) \tag{2a}$$

where δ is a Kroneker delta operator. Similarly, the stiffness of the foundation provided by wires oriented in the y direction is

$$k_1(x, y) = \frac{T}{(b-y)y}\delta(x - x_i) \tag{2b}$$

The plate being thin, its stability can be analyzed by a classical plate theory. Accordingly, the equation of equilibrium is

$$D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} + N_x w_{,xx} + N_y w_{,yy} + [k_1(x, y) + k_2(x, y)]w = 0 \tag{3}$$

Note that the terms proportional to the elements of the stiffness matrix associated with the bending–twisting coupling, that is, D_{16} and D_{26}, do not appear in Eq. (3) because these elements are negligible compared to D_{ij} $(ij \neq 16, 26)$ in multilayered symmetric laminates. In-plane stress resultants applied to the plate, that is, N_x and N_y, remain constant as long as the problem is geometrically linear. In the case where the plate is subject to a uniform temperature, these terms are also readily available. Tensile stresses in the wires do not affect in-plane stress resultants because the wires freely slide along the sleeves and as explained above their reactions are not transferred to the plate.

The boundary conditions that are not affected by the presence of wires are satisfied by representing the mode shape of buckling by the double Fourier series

$$w = \sum_{m,n} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{4}$$

The substitution of Eqs. (1), (2), and (4) into Eq. (3) and the application of the Galerkin procedure yield a system of coupled algebraic equations

$$\sum_{r,s} K_{rsmn} W_{rs} + F_{mn} W_{mn} = N_x \left(\frac{m\pi}{a}\right)^2 W_{mn} + N_y \left(\frac{n\pi}{b}\right)^2 W_{mn} \tag{5}$$

where

$$K_{rsmn} = \frac{4}{ab}\int_{a_1}^{a_2}\int_{b_1}^{b_2} [k_1(x, y) + k_2(x, y)] \sin\frac{r\pi x}{a} \sin\frac{m\pi x}{a} \sin\frac{s\pi y}{b} \sin\frac{n\pi y}{b}\,\mathrm{d}y\,\mathrm{d}x$$

$$F_{mn} = D_{11}\left(\frac{m\pi}{a}\right)^4 + 2(D_{12} + 2D_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4 \tag{6}$$

The integral in Eq. (6) can be evaluated [11]:

$$K_{rsmn} = \frac{4T}{ab}\left\{\sum_i \sin\frac{r\pi x_i}{a} \sin\frac{m\pi x_i}{a} \operatorname{Re}\int_{b_1}^{b_2} \frac{\sin\frac{s\pi y}{b} \sin\frac{s\pi y}{b}}{(b-y)y}\,\mathrm{d}y + \sum_j \sin\frac{s\pi y_j}{b} \sin\frac{n\pi y_j}{b} \operatorname{Re}\int_{a_1}^{a_2} \frac{\sin\frac{r\pi x}{a} \sin\frac{m\pi x}{a}}{(a-x)x}\,\mathrm{d}x\right\} \tag{7}$$

where the integrals are complex functions including sine and cosine integral functions that are available using [11] and Re identifies the real part of the corresponding integral.

In the case where the analysis is conducted using a one-term approximation for the buckling mode shape, the integrals in Eq. (7) are real and simplified. For example, if $r = m = 1$,

$$\int_{a_1}^{a_2} \frac{\sin\frac{r\pi x}{a}\sin\frac{m\pi x}{a}}{(a-x)x}\,\mathrm{d}x = \frac{\log a_2 - \log a_1}{2a} - \frac{\log(a_2-a) - \log(a_1-a)}{2a} - \frac{1}{2a}\left[Ci\left(\frac{2\pi a_2}{a}\right) - Ci\left(\frac{2\pi a_1}{a}\right) + Ci\left(\frac{2\pi a_2}{a} - 2\pi\right) - Ci\left(\frac{2\pi a_1}{a} - 2\pi\right)\right] \tag{8}$$

where Ci are cosine integrals.

Consider now a large aspect ratio plate where $b \gg a$ that is supported by equidistant SMA wires in sleeves oriented along the short edges and subject to an applied load N_x. It can be assumed that the mode shape of buckling of the plate is a cylindrical surface. Then the Rayleigh–Ritz solution is obtained by approximating the buckling mode by

$$w = \frac{4W(a-x)x}{a^2} \tag{9}$$

The buckling load is obtained by substitution of Eq. (9) in the total energy per unit width of the plate:

$$\Pi = \frac{1}{2}\int_0^a \left[D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + k_2(x)w^2 - N_x\left(\frac{\partial w}{\partial x}\right)^2\right]\mathrm{d}x \tag{10}$$

The application of the Rayleigh–Ritz method yields the following result for the case where SMA wires support the plate throughout the entire span $0 < x < a$:

$$N_{x,\mathrm{cr}} = \frac{12D_{11}}{a^2} + \frac{T}{2l_y} \tag{11}$$

It is possible to estimate inaccuracy introduced by the one-dimensional analysis. The buckling load of the plate without SMA wires given by the first term of Eq. (11) is 20% higher than the actual buckling load for such a plate. Therefore, a one-term Rayleigh–Ritz analysis can be used only for a qualitative estimate of the effectiveness of SMA wires. The expansion of the solution by the Rayleigh–Ritz method to the case of a finite aspect ratio simply supported plate buckling into a single half-wave in both the x and y directions could be easily obtained. This extrapolation is omitted here for brevity.

III. SMA Wires Connected to the Plate at the Center

An alternative approach to design can employ SMA wires in the sleeves, or even without sleeves, connected to the plate at a number of points, rather than along a continuous line (Fig. 2). For example, if two mutually perpendicular wires are supporting a rectangular plate at the center, their reaction is

$$R_{w1} \approx 4T\left(\frac{w(a/2, b/2)}{a} + \frac{w(a/2, b/2)}{b}\right) \tag{12}$$

The first term in the brackets on the right side of Eq. (12) follows from the consideration of Fig. 2b. The second term is obtained by analogy.

Consider now two mutually perpendicular systems of wires supporting the plate at a number of points. If the recovery forces in the wires oriented in the x and y directions are denoted by T_x and T_y, respectively, the reaction of wires at a support point (x_i, y_j) is

$$R'_{ij} = T_x\left[\frac{W(x_i, y_j) - W(x_{i-1}, y_j)}{s_{(i-1)ij}} + \frac{W(x_i, y_j) - W(x_{i+1}, y_j)}{s_{i(i+1)j}}\right] + T_y\left[\frac{W(x_i, y_j) - W(x_i, y_{j-1})}{s_{i(j-1)j}} + \frac{W(x_i, y_j) - W(x_i, y_{j+1})}{s_{ij(j+1)}}\right] \tag{13}$$

where the notation for the spacing between the points is evident from Fig. 2a.

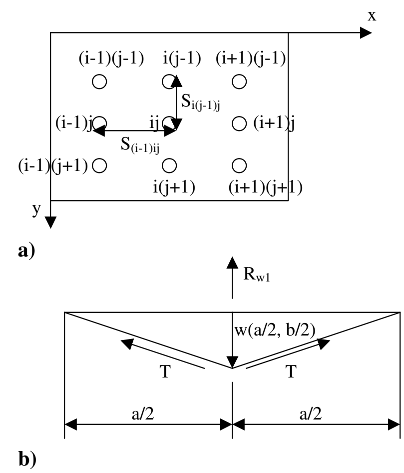

Fig. 2 a) Selected SMA support points of the plate; b) cross section $y = b/2$ of a plate supported by a couple of wires at the center. The analysis of similar force and geometric triangles explains Eq. (12).

Equation (13) can be conveniently written in the form

$$R'_{ij} = f_{ij}W(x_i, y_j) + f_{(i-1)j}W(x_{(i-1)}, y_j) + f_{(i+1)j}W(x_{(i+1)}, y_j) + f_{i(j-1)}W(x_i, y_{(j-1)}) + f_{i(j+1)}W(x_i, y_{(j+1)}) \tag{14}$$

where f_{ij} are coefficients that are easily determined.

If the mode shape of buckling is represented by series (4), the reaction at a support point (x_i, y_j) can be expressed in terms of the amplitudes of the terms, that is,

$$R'_{ij} = \sum_{m,n} S_{ijmn}W_{mn} \tag{15}$$

with S_{ijmn} being functions of the recovery force in the wires and the location of the contact points.

The reactions given by Eq. (15) can be further represented in terms of a double Fourier series

$$p(w) = \sum_{m,n}\sum_{i,j} c_{ijmn}W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad c_{mnij} = \frac{4}{ab}S_{ijmn}\sin\frac{m\pi x_i}{a}\sin\frac{n\pi y_j}{b} \tag{16}$$

where $p(w)$ is an equivalent counterpressure that replaces concentrated reactions applied at points (x_i, y_j) with a continuous function of in-plane coordinates.

The equation of equilibrium becomes

$$D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} + N_x w_{,xx} + N_y w_{,yy} + p(w) = 0 \tag{17}$$

The substitution of the buckling mode shape (4) into Eq. (17) and the Galerkin procedure yield uncoupled buckling equations:

$$F_{mn} + \frac{4}{ab}\sum_{i,j} S_{ijmn}\sin\frac{m\pi x_i}{a}\sin\frac{n\pi y_j}{b} = N_x\left(\frac{m\pi}{a}\right)^2 + N_y\left(\frac{n\pi}{b}\right)^2 \tag{18}$$

IV. Numerical Examples

Cross-ply symmetrically laminated plates considered in the examples were manufactured from AS4/3501-6 graphite epoxy. In all examples the ratio of the buckling load in the presence of activated SMA wires to the counterpart for the same plates without wires, that is, $R = N_{x,\mathrm{cr}}(T)/N_{x,\mathrm{cr}}(T = 0)$, is shown as a function of the recovery force. In the example reflected in Fig. 3 large aspect ratio plates subject to compressive stress resultant N_x were supported by equidistant SMA wires with the spacing l_y oriented along short edges

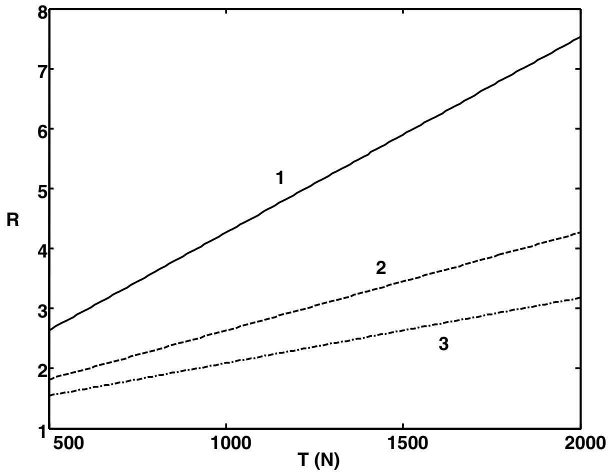

Fig. 3 The effect of a system of closely spaced SMA wires oriented in the x direction and supporting a large aspect ratio, simply supported, symmetrically laminated cross-ply AS4/3501-6 plate ($a = 1.0$ m, $h = 2.0$ mm) on the buckling load N_x. Case 1: $l_y = 0.25$ m; case 2: $l_y = 0.50$ m; case 3: $l_y = 0.75$ m.

$y = 0$, $y = b$. The examples in Figs. 4–7 elucidate the effect of two mutually perpendicular SMA wires supporting the center of the plate. In the cases shown in Figs. 4 and 5 the plate is subject to uniaxial compression N_x, while in Figs. 6 and 7 the plates undergoes biaxial compression. The force in the wires was estimated based on the recovery stress of 220 MPa recorded by Cross for nitinol [12]. For example, the recovery forces in 3- and 5-mm diam wires are equal to $T = 1.56$ kN and $T = 4.32$ kN, respectively.

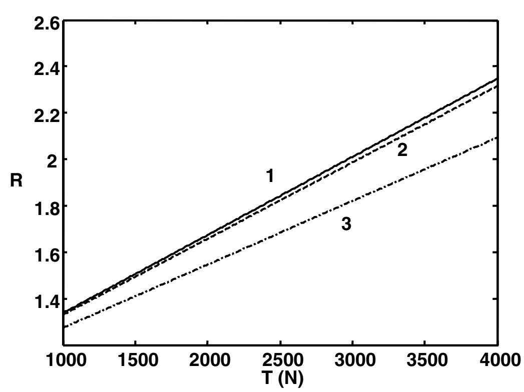

Fig. 4 The effect of SMA wires supporting the center of a simply supported, symmetrically laminated cross-ply AS4/3501-6 plate ($a = 1.0$ m, $h = 4.0$ mm) on the buckling load N_x. Case 1: $a/b = 0.75$; case 2: $a/b = 1.0$; case 3: $a/b = 1.25$.

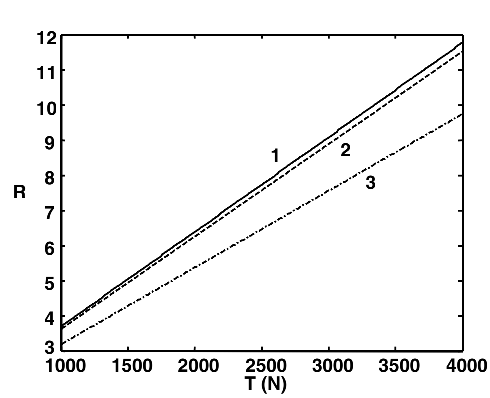

Fig. 5 The effect of SMA wires supporting the center of a simply supported, symmetrically laminated cross-ply AS4/3501-6 plate ($a = 1.0$ m, $h = 2.0$ mm) on the buckling load N_x. Case 1: $a/b = 0.75$; case 2: $a/b = 1.0$; case 3: $a/b = 1.25$.

As follows from Fig. 3, the buckling load of a large aspect ratio plate can be significantly increased using a system of SMA wires of a relatively small diameter, even if the spacing of these wires is quite large. The stability can further be increased by either increasing the wire diameter (larger recovery force) or reducing the spacing.

The results shown in Figs. 4 and 5 illustrate that a couple of SMA wires supporting the center of a finite aspect ratio plate is sufficient for a large increase in the buckling load. More flexible plates are easier controlled using the same system of wires (compare Figs. 4 and 5). The effectiveness of the support provided by SMA wires decreases in more narrow plates where the overall applied compressive force is smaller (compare case 3 to cases 1 and 2).

The effect of biaxial compression is reflected in Figs. 6 and 7. As follows from these figures, the buckling combination of biaxial compressive loads can be increased several times supporting the center of a finite aspect ratio plate with a couple of SMA wires. A representative comparison with the results generated by a finite element method shown in Fig. 6 confirms the accuracy of the analytical solution and the feasibility of increasing buckling capacity of plates using the proposed approach. The finite element analysis was conducted modeling the plate by thin 2-D rectangular elements. The effect of the SMA wires was modeled by a linear spring with the stiffness evaluated according to Eq. (12).

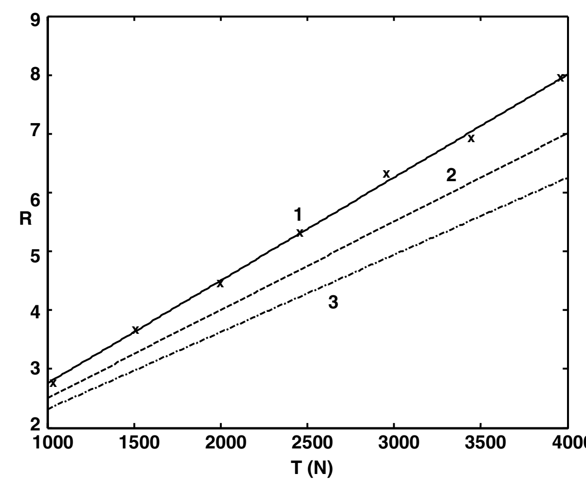

Fig. 6 The effect of SMA wires supporting the center of a simply supported, square, symmetrically laminated cross-ply AS4/3501-6 plate ($a = 1.0$ m, $h = 2.0$ mm) on the biaxial buckling load. Case 1: $N_y/N_x = 0.5$; case 2: $N_y/N_x = 0.75$; case 3: $N_y/N_x = 1.0$. Finite element method (FEA) results are shown by x for case 1.

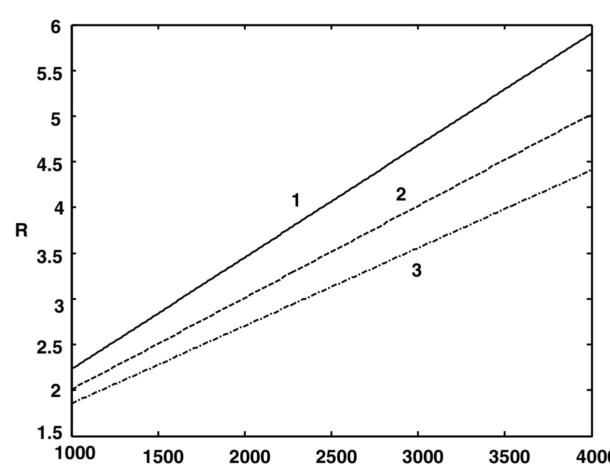

Fig. 7 The effect of SMA wires supporting the center of a simply supported, symmetrically laminated cross-ply AS4/3501-6 plate ($a = 1.0$ m, $b = 0.8$ m, $h = 20$ mm) on the biaxial buckling load. Case 1: $N_y/N_x = 0.5$; case 2: $N_y/N_x = 0.75$; case 3: $N_y/N_x = 1.0$.

Although the previous examples clearly illustrate the benefits of buckling control of composite plates using SMA wires, it is necessary to estimate the additional associated weight. The weight of the sleeves can be assumed negligible so that it is sufficient to compare the weight of the SMA wires to that of the supported structure. The weight of the SMA wires can be found by relating the prescribed recovery force to the cross section of the wire assuming the recovery stress of 220 MPa. A casual inspection illustrates that by using SMA wires weighing just 5% of the overall weight of the structure one can triple or quadruple the buckling load.

V. Conclusions

This paper illustrates the feasibility of buckling control of composite plates using SMA wires located outside the plate that provide either continuous or point support. The outside placement of the wires simplifies the manufacture and activation process and eliminates or reduces thermal effects in the plate. The buckling load can be increased several times using wires that add 5% or even less to the total weight of the plate. This illustrates that the proposed methods that were shown effective for vibration control in the previous work of the author can also be successfully applied to enhance stability of composite and isotropic plates. The present solution may further be extended to consider the advantages of a nonuniform spacing of wires and a strategic distribution of support points with different wires leading to an optimization of the design.

Acknowledgments

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